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Approximation theorems in metric spaces and functionals strictly subordinated to convergent series

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Dedicated to the memory of V.V. Fedorchuk, an outstanding mathematician and a wonderful person

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ABSTRACT

Some new results are presented concerning the search and approximation of solutions of equations in metric spaces using functionals strictly subordinated to a convergent series and functionals compatible with a convergent series. These classes of functionals were earlier introduced by the author.

The problems of search and approximation of solutions of equations in metric spaces are considered.

The fixed point search and approximation problems of a given self-mapping of a metric space go back to classical Banach contraction principle and its diverse generalizations. Many classical results in this area are collected in the remarkable book [1] (see also [3,4]). Let us list some of them. For example, the fixed point theorem in a complete metric space by J. Matkowski [2], in which the classical Banach metric inequality

$$d\big(F(x), F(y)\big) \le k \cdot d(x, y) \tag{$*$}$$

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was replaced by the inequality $d(F(x), F(y)) \leq \varphi(d(x, y))$ with a nondecreasing function $\varphi(t)$ such that $\varphi^n(t) \to 0$ for any $t > 0$. Let us also mention the result [1, Ch. 1, §1, (5.3)] on general contractions in a complete metric space, where the coefficient k in $(*)$ is replaced by a function $\alpha(x, y)$ with some special properties. In addition, the result by J. Weissinger [5] (see also [1, Ch. 1, §1, (A.5)]) should be mentioned where Banach constant k is replaced by terms of a positive convergent series $\sum_{n=1}^{\infty} a_n$. More exactly, instead of $(*)$, the following inequality was suggested: $d(F^n(x), F^n(y)) \leq a_n \cdot d(x, y)$. In such a way, the geometrical progression with the common ratio k arising in the iteration process in Banach contraction principle was replaced by an arbitrary positive convergent series. A further generalization of Banach principle was obtained by F.E. Browder [6] (see also [1, Ch. 1, §1, (B.2)]) in his famous fixed point theorem, also in a complete metric space, where he considered φ-contractive mapping, that is he used the inequality $d(F(x), F(y)) \leq \varphi(d(x, y))$, with a nondecreasing function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying the following conditions: (i) $\varphi(t) < t$ for any $t > 0$, (ii) φ is right continuous. The survey of J. Jachymski [7] is devoted to this remarkable Browder theorem, some its generalizations and open questions around it. Of course, we should also mention the well-known Nadler theorem [8], which generalizes Banach contraction principle to the case of multivalued mappings.

All the listed results as well as many others are devoted to the fixed point existence and uniqueness problems. As for analogous coincidence problems, when n $(n \geq 2)$ mappings transform one metric space to another, it is not always possible to reduce them to the fixed point problems. Therefore, different approximation schemes are needed for solving coincidence problems.

To solve some versions of coincidence problems, in this paper we use auxiliary functionals with special properties. In particular, we take advantage of so called functionals strictly subordinated to convergent series (it should be noted that in [15] such functionals were named strictly subjected to convergent series) and functionals compatible with convergent series, which were introduced by the author (see [15,14]). On the one hand, functionals strictly subordinated to convergent series represent some generalizations of (α, β)-search, generally (α, β)-search and almost exactly (α, β)-search functionals $(0 < \beta < \alpha)$ (see [10–12,15,14] for definitions, examples and comparisons). On the other hand, functionals strictly subordinated to convergent series are a particular case of functionals subordinated to convergent series (see [17,18] for the definition and some applications).

It should be noted that the idea of using a majorizing convergent series in the approximation process in some different special form was exploited in [16] with regard to the fixed point existence and approximation problem in the case of multivalued contractions.

Theorems 1–4 presented here were announced in [14]. The proofs of Theorems 1, 2 were given in [15].

Let (X, ρ) be a metric space, $\mathbb{R}_+ = \{t \in \mathbb{R} \mid t \geq 0\}$ be the set of nonnegative real numbers, and let $P(A)$ stand for the totality of all nonempty subsets of a set A. Let also the following convergent series with monotonically decreasing positive terms be fixed.

$$\sum_{j=1}^{\infty} c_j < \infty, \quad 0 < c_{n+1} < c_n, \; n \in \mathbb{N}. \tag{1}$$

Denote the sum of the residual series of (1) by $S_k = \sum_{j=k}^{\infty} c_n$, $k \in \mathbb{N}$.

Definition 1. ([15,14]) A multivalued nonnegative functional $\varphi : X \to P(\mathbb{R}_+)$ is said to be *strictly subordinated to series* (1) on a metric space (X, ρ), if the following conditions are fulfilled for its graph $Gr(\varphi)$:

1) if for a pair $(x, t) \in Gr(\varphi)$ it is true that $t > c_1$, then there exist a pair $(x', t') \in Gr(\varphi)$ and a number k, $k \in \mathbb{N}$, such that $\rho(x, x') \leq t$ and $t' \leq c_k$;

2) if for a pair $(x, t) \in Gr(\varphi)$ it is true that $t \leq c_k$ for some $k \in \mathbb{N}$, then there exists a pair $(x', t') \in Gr(\varphi)$ such that $\rho(x, x') \leq t$ and $t' \leq \frac{t}{c_k} c_{k+1}$. □

Given metric spaces (X, ρ), (Y, d), everywhere below we consider in the space $X \times Y$ the metric D, where $D((x, y), (x', y')) = \rho(x, x') + d(y, y')$ for any $x, x' \in X$, $y, y' \in Y$.

Let us recall some definitions more (see also [10,15,14]).

The graph $Gr(\varphi)$ of a multivalued functional $\varphi : X \to P(\mathbb{R}_+)$ is called $\{0\}$-*complete* if any fundamental sequence $\{(x_m, t_m)\}_{m=0,1,...} \subseteq Gr(\varphi)$, with $t_m \to 0$, converges to some pair $(\xi, 0) \in Gr(\varphi)$, that is $\xi \in Nil(\varphi) = \{x \in X \mid 0 \in \varphi(x)\}$.

The graph $Gr(\varphi)$ of a functional φ is called $\{0\}$-*closed*, if any its limit element of the form $(\xi, 0)$ belongs to it.

A very convenient and short term *cascade* was suggested by D.V. Anosov for a discrete dynamic systems (see his article “Cascade” in [9]). As well as in some our previous papers, below we use the term “multicascade” for multivalued discrete dynamic systems. So, *a multicascade* on X is a multivalued discrete dynamic system with the phase space X and the additive translation semigroup $(\mathbb{Z}_+, +)$ (where $\mathbb{Z}_+ = \{0, 1, 2, ...\}$), that is an arbitrary (multivalued) action of that semigroup on X. In other words, say a multicascade is given on X if some (multivalued) mapping $\mathcal{G} : X \to P(X)$ is given, which nonnegative iterations generate the semigroup $\{\mathcal{G}^n\}_{n=0,1,...}$, where $\mathcal{G}^0 := \mathrm{id}_X$ is the identity mapping. The last semigroup clearly represents the semigroup $(\mathbb{Z}_+, +)$.

The mapping $\mathcal{G} = \mathcal{G}^1 : X \to P(X)$ itself representing the generating element $1 \in \mathbb{Z}_+$ is called *the generator* of the multicascade.

A trajectory of the multicascade is any sequence $\{x_n\}_{n=0,1,2,...}$, where $x_{n+1} \in \mathcal{G}(x_n)$, $n = 0, 1, 2, ...$. So, nonnegative iterations of the generator $\mathcal{G}$, being applied to an arbitrary point $x \in X$, form trajectories starting from the point x.

The limits of all trajectories (if exist) form *the limit set* of the multicascade.

We say that a limit point $\xi \in X$ of a given multicascade on X *corresponds to a (starting) point* $x_0 \in X$ if there is a trajectory of the multicascade starting from x_0 and converging to ξ.

The graph of a multivalued mapping $F : X \to C(Y)$, where $C(Y)$ stands for the totality of nonempty closed subsets of Y, is called H-*complete* for some closed subspace $H \subset Y$, if any fundamental sequence $\{(x_m, y_m)\}_{m=0,1,...} \subseteq Gr(F)$, with $d(y_m, H) \to 0$, converges to some element of the graph $Gr(F)$. We call the set $F^{-1}(H) = \{x \in X \mid F(x) \cap H \neq \emptyset\}$ *the full preimage* of a given closed subspace H under the action of a given multivalued mapping $F : X \to C(Y)$. Below we denote $\Delta_n(H) = \{\tilde{y} = (y, ..., y) \in Y^n \mid y \in H\}$.

The next theorem demonstrates properties of functionals strictly subordinated to convergent series. Its proof is rather standart and was given in [15]. Nevertheless, for the completeness of the exposition, we repeat it below.

Theorem 1. *([15,14]) Let a multivalued functional $\varphi : X \to P(\mathbb{R}_+)$ be strictly subordinated to series (1) on the space X and either the graph $Gr(\varphi)$ be $\{0\}$-complete or X be complete and the graph $Gr(\varphi)$ be $\{0\}$-closed. Then, a multicascade is defined on $Gr(\varphi)$ with nonempty limit set $\mathcal{A} \subseteq Gr(\varphi)$, which projection $\mathcal{A}_X$ onto X is equal to $Nil(\varphi) = \{x \in X \mid 0 \in \varphi(x)\}$. In addition, if for a pair $(x_0, t_0) \in Gr(\varphi)$ it is true that $t_0 \leq c_{k_0} \cdot \min\{1, \frac{r}{S_{k_0}}\}$, for some numbers $k_0 \in \mathbb{N}$ and $r > 0$, then there is a limit pair $(\xi, 0) \in \mathcal{A}$ corresponding to (x_0, t_0), such that $\xi \in \mathcal{A}_X \cap U(x_0, r)$.*

Proof. The existence of the required multicascade is proved similarly to the proving of [11, Theorem 1] (see also corresponding theorems in [10]). Indeed, as the functional φ is strictly subordinated to series (1), the set $\mathcal{G}((x, t)) = \{(x', t') \in Gr(\varphi) \mid (x', t')$ meets conditions 1), 2) of Definition 1$\}$ is defined (and nonempty) for any pair $(x, t) \in Gr(\varphi)$. It is easy to see that the multivalued mapping $\mathcal{G} : Gr(\varphi) \to P(Gr(\varphi))$ is a generator of a multicascade on $Gr(\varphi)$. Trajectories of this multicascade are convergent sequences of the form of $\{(x_m, t_m)\}_{m=0,1,...} \subseteq Gr(\varphi)$ starting from an arbitrary pair $(x_0, t_0) \in Gr(\varphi)$. If $t_0 \leq c_{k_0}$, then the following inequalities hold

$$t_m \le \frac{t_0}{c_{k_0}} c_{k_0+m}, \rho(x_m, x_{m+1}) \le t_m, \quad m \in \mathbb{N} \cup \{0\}. \tag{2}$$

To show this, suppose $t_0 \le c_{k_0}$. Then $t_1 \le \frac{t_0}{c_{k_0}} c_{k_0+1} \le c_{k_0+1}$,

$$t_2 \le \frac{t_1}{c_{k_0+1}} c_{k_0+2} \le \frac{t_0}{c_{k_0}} c_{k_0+2}, ..., t_m \le \frac{t_0}{c_{k_0}} c_{k_0+m}.$$

And, if $t_0 > c_1$, then there exist a pair $(x_1, t_1) \in Gr(\varphi)$ and a number $k_1 \in \mathbb{N}$, such that $\rho(x_0, x_1) \le t_0$, $t_1 \le c_{k_1}$. Consequently, similarly to (2), we obtain

$$t_{m+1} \le t_1 \frac{c_{k_1+m}}{c_{k_1}}, \quad m \in \{0\} \cup \mathbb{N}. \tag{3}$$

Using inequalities (2) or (3), we obtain the following estimations of the distance between the starting point x_0 and any corresponding to it limit point ξ. If $t_0 \le c_{k_0}$ for some $k_0 \in \mathbb{N}$, then

$$\rho(x_0, \xi) \le \sum_{m=0}^{\infty} \rho(x_m, x_{m+1}) \le \sum_{m=0}^{\infty} t_m \le \frac{t_0}{c_{k_0}} \sum_{m=0}^{\infty} c_{k_0+m} = \frac{t_0 S_{k_0}}{c_{k_0}}.$$

And if $t_0 > c_1$, then

$$\rho(x_0, \xi) \le \rho(x_0, x_1) + \sum_{m=1}^{\infty} \rho(x_m, x_{m+1}) \le t_0 + \sum_{m=1}^{\infty} t_m \le t_0 + \frac{t_1}{c_{k_1}} \sum_{m=0}^{\infty} c_{k_1+m} = t_0 + \frac{t_1 S_{k_1}}{c_{k_1}}.$$

It follows that if t_0 satisfies the estimation given in the condition of the theorem, then $\rho(x_0, \xi) \le r$, that is $\xi \in U(x_0, r)$, which was to be proved. $\square$

Let us apply Theorem 1 to the search and approximation problem of the *set of common preimages* of a closed subspace $H \subset Y$ under actions of a given finite set of multivalued mappings $F_1, ..., F_n : X \to C(Y)$. We mean the set

$$Coin_H(F_1, ..., F_n) := \left\{ x \in X \;\middle|\; H \cap \left(\bigcap_{i=1}^{n} F_i(x) \right) \ne \emptyset \right\}.$$

Let (X, ρ), (Y, d) be metric spaces, a multivalued mapping $Q : X \to P(Y)$ and a functional $\psi : Gr(Q) \to \mathbb{R}_+$ on the graph $Gr(Q)$ of the mapping Q be given. Below we shall need to consider the following strengthened version of the condition for a functional to be strictly subordinated to a series. Suppose there is a number $\gamma > 0$ such that for any pair $(x, y) \in Gr(Q)$ there exists a pair $(x', y') \in Gr(Q)$ such that the following conditions hold:

$$\begin{gathered} \rho(x, x') \le \psi(x), \qquad D(y, y') \le \gamma \cdot \psi(x), \\ (\exists k \in \mathbb{N},\ \psi(x) \le c_k) \quad \Longrightarrow \quad \left(\psi(x') \le \frac{\psi(x)}{c_k} \cdot c_{k+1} \right), \\ (\psi(x) > c_1) \quad \Longrightarrow \quad (\psi(x') \le c_1). \end{gathered} \tag{4}$$

Theorem 2. *([15,14]) Let multivalued mappings $F_1, F_2, ..., F_n : X \to C(Y)$ be given, and $H \subseteq Y$ be a closed subspace. Denote $F = F_1 \times F_2 \times ... \times F_n : X \to C(Y^n)$. Suppose that the graph $Gr(F)$ is $\Delta_n(H)$-closed, and one of the graphs $Gr(F_i)$, $i = 1, ..., n$, is H-complete. Let for some $\gamma > 0$ the functional $\psi : Gr(F) \to \mathbb{R}_+$, $\psi(x, y) = D(y, \Delta_n(H))$ meet conditions (4).*

Then a multicascade is defined on $Gr(F)$ with limit set $\mathcal{A} \neq \emptyset$, which projection onto X is equal to $Coin_H(F_1, ..., F_n)$. In addition, if $\psi(x_0, y_0) \leq c_{k_0} \cdot \min\{1, \frac{r}{S_{k_0}}\}$ for a pair $(x_0, y_0) \in Gr(F)$, then every trajectory of the multicascade beginning from (x_0, y_0) converges to some $(\xi, \eta) \in Gr(F)$, where $\rho(x_0, \xi) \leq r$ and $d(y_0, \eta) \leq \gamma r$. □

In the case of $H = Y$ and $n \geq 2$, Theorem 2 solves the problem of local search for the coincidence set of a given collection of mappings. In the case of $H = \{c\}$, $c \in Y$, this theorem enables one to determine the common roots of the mappings $F_1, ..., F_n$ corresponding to their common value c.

Let us now consider the problem of local search for roots of an equation of the form of $f(x) = c$, where $f : X \to Y$ is a mapping between metric spaces, in a somewhat different setting.

Definition 2. ([15]) A functional $\lambda : \mathbb{R}_+ \to \mathbb{R}_+$ is called *compatible with series* (1) if $\lambda(t) \leq \frac{c_{k+1}}{c_k} \cdot t$ for $t \in (c_{k+1}; c_k]$, $k \in \mathbb{N}$, and $\lambda(t) \leq c_1$ for $t > c_1$.

Let a mapping $f : X \to Y$ and a functional $\varphi : X \to \mathbb{R}_+$ be given. In the next theorem we shall need to consider the situation when for some $\gamma > 0$, and some functions $\mu, \nu : \mathbb{R}_+ \to \mathbb{R}_+$ the following conditions are fulfilled:

$$\begin{gathered}\text{for any } x \in X \text{ there exists } x' \in X \\ \text{such that } \rho(x, x') \leq \mu(\varphi(x)), \\ d(f(x), f(x')) \leq \gamma \cdot \varphi(x), \qquad \varphi(x') \leq \nu(\rho(x, x'))\end{gathered} \tag{5}$$

Theorem 3. *([15]) Let a mapping $f : X \to Y$ be given. Let $\mu, \nu : \mathbb{R}_+ \to \mathbb{R}_+$ be nondecreasing commuting functions, positive on $(0; +\infty)$, with $\mu(0) = \nu(0) = 0$. Suppose that the function $\lambda(t) = \mu\nu(t) = \nu\mu(t)$ is compatible with series (1). Let also $\gamma > 0$ and $c \in Y$ be fixed, and the graph $Gr(f)$ be $\{c\}$-complete. Suppose that the functional $\varphi : X \to \mathbb{R}_+$, $\varphi(x) = d(f(x), c)$, meets conditions (5) with respect to μ, ν. Then, on X, a multicascade with nonempty limit set $Nil(\varphi) = \{x \in X \mid f(x) = c\}$ is defined. In addition, if $\mu(\varphi(x_0)) \leq c_{k_0} \cdot \min\{1, \frac{R}{S_{k_0}}\}$ for some $x_0 \in X, R > 0, k_0 \in \mathbb{N}$, then any trajectory of the multicascade beginning from x_0 converges to a solution ξ (generally speaking, not unique) of the equation $f(x) = c$, and $\rho(x_0, \xi) \leq R$.*

Proof. Take any $x_0 \in X$. Then there exists $x_1 \in X$ such that $\rho_0 = \rho(x_0, x_1) \leq \mu(\varphi(x_0))$, and $\varphi(x_1) \leq \nu(\rho(x_0, x_1)) \leq \nu(\mu(\varphi(x_0)))$. So, $\varphi(x_1) \leq \nu(\mu(\varphi(x_0)))$. Proceeding in such a way, one can construct a sequence $\{x_m\}_{n=0,1,...}$ satisfying the following conditions

$$\rho_m \leq \mu\nu(\rho_{m-1}) \leq (\mu\nu)^m(\rho_0), \tag{6}$$

$$\varphi_m \leq \nu\mu(\varphi_{m-1}) \leq (\nu\mu)^m(\varphi_0), \tag{7}$$

where $\rho_m = \rho(x_m, x_{m+1}), \varphi_m = \varphi(x_m), m \in \{0\} \cup \mathbb{N}$. Let $\rho_0 \in (c_{k_0+1}; c_{k_0}]$ and $\varphi(x_0) \in (c_{q_0+1}; c_{q_0}]$ for some $k_0, q_0 \in \mathbb{N}$. Then, as the functional $\lambda(t) = \mu\nu(t) = \nu\mu(t)$ is compatible with series (1), we obtain from (6), (7) the following estimations

$$\rho_m \leq \frac{\rho_0}{c_{k_0}} \cdot c_{k_0+m}, \tag{8}$$

$$\varphi_m \leq \frac{\varphi_0}{c_{q_0}} \cdot c_{q_0+m}. \tag{9}$$

Inequalities (8), (9) imply that the sequence $\{x_m\}_{n=0,1,...}$ is fundamental, and $\varphi(x_m) \to 0$ when $m \to \infty$. Moreover, as $d_m = d(f(x_m), f(x_{m+1})) \leq \gamma \cdot \varphi(x_m) \leq \gamma \cdot \frac{\varphi_0}{c_{q_0}} \cdot c_{q_0+m}$, it follows that the sequence

$\{f(x_m)\}_{n=0,1,\dots}$ is also fundamental. Hence the sequence $\{(x_m, f(x_m))\}_{n=0,1,\dots}$ is fundamental. Finally, as graph $Gr(f)$ is $\{c\}$-complete, the sequence $\{(x_m, f(x_m))\}_{n=0,1,\dots}$ converges to some pair $(\xi,\eta) \in Gr(f)$. It follows from the continuity of the metric d, that $d(\eta,c) = \lim_{m\to\infty} d(f(x_m),c) = \lim_{m\to\infty}\varphi(x_m) = 0$. Consequently $\eta = f(\xi) = c$. So, ξ is a solution of the equation $f(x) = c$. $\square$

Below we shall need the following two definitions.

Definition 3. ([15]) Let subsets $U \subseteq X, W \subseteq Y$, a functional $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$, with $\varphi(0) = 0$, and a collection $\mathcal{U} = \{(x,r) \subseteq X \times \mathbb{R}_+\}$, with $B(x,r) \subseteq U$ for any $(x,r) \in \mathcal{U}$, be given. A mapping $F : X \to Y$ is said to be *conditionally $\varphi(t)$-covering the set W relative to U on the collection $\mathcal{U}$* if the inclusion $B(F(x),\varphi(r)) \cap W \cap F(U) \subseteq F(B(x,r))$ holds for any $(x,r) \in \mathcal{U}$.

Definition 4. ([15]) A mapping $f : X \to Y$ of metric spaces is said to be $\psi(t)$-*contractive*, if $d(f(x),f(y)) \le \psi(\rho(x,y))$ where $\psi : \mathbb{R}_+ \to \mathbb{R}_+$, $\psi(0) = 0$, $x,y \in X$.

Given a mapping $F : X \times X \to Y$ and $z \in Y$, consider the search and approximation problem of solutions of the equation $F(x,x) = z$.

We shall need to consider the following situation. Let X be a complete metric space. Let $x_0 \in X$, $z_0 \in Y$, $R > 0$, a mapping $F : B(x_0,R)\times B(x_0,R) \to Y$ and nondecreasing commuting functions $\varphi,\psi : \mathbb{R}_+ \to \mathbb{R}_+$ be given, with $\varphi(0) = \lim_{t\to+0}\varphi(t) = \psi(0) = \lim_{t\to+0}\psi(t) = 0$, and let φ be invertible. For any $y \in B(x_0,R)$, denote $\mathcal{B}(y) = \{(y,r) \mid 0 \le r \le R - \rho(x_0,y)\}$.

We shall also need the following conditions concerning the convergent (at a given point $t > 0$) functional series $\sum_{j=0}^{\infty}(\varphi^{-1}\psi)^j(\varphi^{-1}(t)) = S(t)$:

$$\begin{aligned}\varphi^{-1}\psi\big(S(t)\big) &\le \sum_{j=1}^{\infty}\big(\varphi^{-1}\psi\big)^j\big(\varphi^{-1}(t)\big),\\ \varphi\big(\big(id - \varphi^{-1}\psi\big)\big(S(t)\big)\big) &\le (\varphi-\psi)\big(S(t)\big).\end{aligned} \tag{10}$$

The following statement represents a development of [13, Theorem 1].

Theorem 4. *([15]) In the situation described above, suppose that following conditions (i)–(iv) hold:*

(i) for any $y \in B(x_0,R)$, the mapping $F(x,y) : B(x_0,R) \to Y$ is conditionally $\varphi(t)$-covering the set $B(F(x_0,y),\varphi(R))$ relative to $B(x_0,R)$ on the ensemble $\mathcal{B}(y)$;

(ii) for any $x \in B(x_0,R)$, the mapping $F(x,y) : B(x_0,R) \to Y$ is $\psi(t)$-contractive with respect to y, $y \in B(x_0,R)$;

(iii) the convergent series $\sum_{j=0}^{\infty}(\varphi^{-1}\psi)^j(\varphi^{-1}(r_0)) = S(r_0)$ meets conditions (10), where $r_0 = d(z_0,F(x_0,x_0))$, and $z_0 \in \bigcap_{\rho(y,x_0)\le S(r_0)} F(B(x_0,R),y)$;

(iv) if $\{x_k\} \subset B(x_0,R)$, $x_k \xrightarrow[k\to\infty]{} x$, and $F(x_k,x) \xrightarrow[k\to\infty]{} y$, then $F(x,x) = y$.

Then, if $S(r_0) \le R$, there exists $\xi \in B(x_0,R)$ such that $F(\xi,\xi) = z_0$.

Proof. The proof scheme suggested here is similar to that of [13, Theorem 1]. At first, let us show that condition (ii), inequality $S(r_0) \le R$, and the condition $z_0 \in \bigcap_{\rho(x,y)\le S(r_0)} F(B(x_0,R),y)$ imply that

$$z_0 \in B\big(F(x_0,y),\varphi(R)\big) \cap F\big(B(x_0,R),y\big) \tag{11}$$

for any y satisfying the inequality $\rho(y,x_0) \le S(r_0)$.

It is necessary to verify that $d(z_0, F(x_0, y)) \le \varphi(R)$. Indeed,

$$d\big(z_0, F(x_0, y)\big) \le d\big(z_0, F(x_0, x_0)\big) + d\big(F(x_0, x_0), F(x_0, y)\big)$$
$$\le r_0 + \psi\big(\rho(x_0, y)\big) \le r_0 + \psi\big(S(r_0)\big),$$

because ψ does not decrease and $\rho(x_0, y) \le S(r_0)$.

Now, we can estimate r_0. Let us note that by condition (iii),

$$\sum_{j=0}^{\infty} \big(\varphi^{-1}\psi\big)^j \big(\varphi^{-1}(r_0)\big) = \varphi^{-1}(r_0) + \sum_{j=1}^{\infty} \big(\varphi^{-1}\psi\big)^j \big(\varphi^{-1}(r_0)\big) = S(r_0).$$

It implies that

$$\varphi^{-1}(r_0) = S(r_0) - \sum_{j=1}^{\infty} \big(\varphi^{-1}\psi\big)^j \big(\varphi^{-1}(r_0)\big)$$
$$\le S(r_0) - \big(\varphi^{-1}\psi\big)\big(S(r_0)\big) = \big(id - \varphi^{-1}\psi\big)\big(S(r_0)\big).$$

Applying φ to the inequality $\varphi^{-1}(r_0) \le (id - \varphi^{-1}\psi)(S(r_0))$, we obtain:

$$r_0 \le \varphi\big(id - \varphi^{-1}\psi\big)\big(S(r_0)\big) \le (\varphi - \psi)\big(S(r_0)\big).$$

Combining the last inequality with the inequality $d(z_0, F(x_0, y)) \le r_0 + \psi(S(r_0))$, one can see that $d(z_0, F(x_0, y)) \le (\varphi - \psi)(S(r_0)) + \psi(S(r_0)) = \varphi(S(r_0)) \le \varphi(R)$, that is $z_0 \in B(F(x_0, y), \varphi(R))$, which was to be verified.

Further, if $y = x_0$, it follows from (11) that

$$z_0 \in B\big(F(x_0, x_0), \varphi(R)\big) \cap F\big(B(x_0, R), x_0\big),$$

and condition (i) turns into the following inclusion:

$$B\big(F(x_0, x_0), r_0\big) \cap B\big(F(x_0, x_0), \varphi(R)\big) \cap F\big(B(x_0, R), x_0\big) \subseteq F\big(B\big(x_0, \varphi^{-1}(r_0)\big), x_0\big).$$

As z_0 belongs to the left part of the last inclusion, it belongs also to the right part of that. Then, we can state that there exists $x_1 \in B(x_0, \varphi^{-1}(r_0))$ such that $F(x_1, x_0) = z_0$.

Let us put $r_1 = \varphi^{-1}\psi(r_0)$ and apply the same reasoning to the pair $(x_1, \varphi^{-1}(r_1))$. It is correct because $\rho(x_0, x_1) + \varphi^{-1}(r_1) \le (id + \varphi^{-1}\psi)(\varphi^{-1}(r_0)) \le S(r_0) \le R$.

Denoting $z_1 = F(x_1, x_1)$, we can see that

$$d(z_0, z_1) = d\big(F(x_1, x_0), F(x_1, x_1)\big) \le \psi\big(\rho(x_1, x_0)\big) \le \psi\varphi^{-1}(r_0) = r_1.$$

Now let us consider condition (i) for the closed ball $B(x_1, \varphi^{-1}(r_1))$. We have the following inclusion

$$B\big(F(x_1, y), r_1\big) \cap B\big(F(x_0, x_1), \varphi(R)\big) \cap F\big(B(x_0, R), y\big)$$
$$\subseteq F\big(B\big(x_1, \varphi^{-1}(r_1)\big), y\big), \quad y \in B(x_0, R).$$

In the case of $y = x_1$, it turns into the following inclusion:

$$B\big(F(x_1, x_1), r_1\big) \cap B\big(F(x_0, x_1), \varphi(R)\big) \cap F\big(B(x_0, R), x_1\big)$$
$$\subseteq F\big(B\big(x_1, \varphi^{-1}(r_1)\big), x_1\big). \tag{12}$$

As $d(z_0, z_1) \le \psi\varphi^{-1}(r_0) = r_1$, it follows that $z_0 \in B(z_1, r_1)$. Combining that with the inclusion (11), we conclude that z_0 belongs to the left part of the inclusion (12). Consequently, $z_0 \in F(B(x_1, \varphi^{-1}(r_1)), x_1)$. It implies that there exists $x_2 \in B(x_1, \varphi^{-1}(r_1))$ such that $F(x_2, x_1) = z_0, \rho(x_2, x_1) \le \varphi^{-1}(d(z_0, z_1)) \le \varphi^{-1}(r_1)$.

Denoting $z_2 = F(x_2, x_2)$, we obtain the following inequalities.

$$\begin{aligned}\rho(x_0, x_2) &\le \rho(x_0, x_1) + \rho(x_1, x_2) \le \varphi^{-1}(r_0) + \varphi^{-1}\psi\big(\varphi^{-1}(r_0)\big) \le S(r_0),\\ r_2 = d(z_0, z_2) &= d\big(F(x_2, x_1), F(x_2, x_2)\big) \le \psi\big(\rho(x_1, x_2)\big)\\ &\le \psi\big(\varphi^{-1}\psi\big)\big(\varphi^{-1}(r_0)\big) = \big(\varphi^{-1}\psi\big)^2(r_0).\end{aligned}$$

Let us proceed with these constructions. If a point x_{m-1} and a radius $r_{m-1} \le (\varphi^{-1}\psi)^{m-1}(r_0)$ are already chosen, one can choose the next point $x_m \in B(x_{m-1}, \varphi^{-1}(r_{m-1}))$ such that $F(x_m, x_{m-1}) = z_0$ and denote $z_m = F(x_m, x_m)$.

So, one can construct sequences $\{x_m\}_{m=0,1,\dots}$, and $\{z_m\}_{m=0,1,\dots}$ with the following properties:

$$\begin{aligned}\rho(x_0, x_m) &\le \sum_{j=0}^{m-1} \big(\varphi^{-1}\psi\big)^j \big(\varphi^{-1}(r_0)\big) \le S(r_0),\\ d(z_m, z_0) &= d\big(F(x_m, x_m), F(x_m, x_{m-1})\big) \le \psi\big(\rho(x_m, x_{m-1})\big)\\ &\le \psi\big(\varphi^{-1}(r_{m-1})\big) = \psi\big(\big(\varphi^{-1}\psi\big)^{m-1}\big(\varphi^{-1}(r_0)\big)\big) = \big(\varphi^{-1}\psi\big)^m(r_0),\\ \rho(x_m, x_{m-1}) &\le \varphi^{-1}(r_{m-1}) = \big(\varphi^{-1}\psi\big)^{m-1}\big(\varphi^{-1}(r_0)\big).\end{aligned}$$

As $(\varphi^{-1}\psi)^{m-1}(\varphi^{-1}(r_0))$ is a member of the convergent series, the sequence $\{x_m\}_{m=0,1,\dots}$ is fundamental, hence (as X is complete) it converges to some point $\xi \in X$, and

$$\rho(x_0, \xi) \le \lim_{m\to\infty} \sum_{j=0}^{m-1} \big(\varphi^{-1}\psi\big)^j \big(\varphi^{-1}(r_0)\big) \le S(r_0) \le R.$$

Let us also note that, as the function φ is right-continuous at zero,

$$d(z_0, z_m) \le r_m = \big(\varphi^{-1}\psi\big)^m(r_0) = \varphi\big(\big(\varphi^{-1}\psi\big)^m\big(\varphi^{-1}(r_0)\big)\big) \xrightarrow[m\to\infty]{} 0.$$

It means that $z_m \xrightarrow[m\to\infty]{} z_0$. In addition, we have

$$d\big(F(x_m, \xi), z_0\big) = d\big(F(x_m, \xi), F(x_m, x_{m-1})\big) \le \psi\big(\rho(\xi, x_{m-1})\big) \xrightarrow[m\to\infty]{} 0,$$

that is $F(x_m, \xi) \xrightarrow[m\to\infty]{} z_0$ when $x_m \xrightarrow[m\to\infty]{} \xi$. Then, by condition (iv), it follows that $F(\xi, \xi) = z_0$, which was to be proved. □

Note that Theorem 4 implies [13, Theorem 1], in the case of linear functions $\varphi(t) = \alpha t$, $\psi(t) = \beta t$, where $0 < \beta < \alpha$.

The next statement follows from Theorem 4 and is an analogue of [13, Theorem 2] concerning the stability of solutions of the equation $F(x, x) = z_0$, under the conditions of Theorem 4, with respect to perturbations of the given mapping F. It was mentioned in [15].

Theorem 5. *Let a sequence of mappings $\{F_k\}_{k=1,2,\dots}$ be given, where $F_k : B(x_0, R) \times B(x_0, R) \to Y, k \in \mathbb{N}$. Suppose conditions (i)–(iv) of Theorem 4 hold for every $F_k, k \in \mathbb{N}$ (with respect to the same functions φ and ψ), and $r_{0k} = d(z_0, F_k(x_0, x_0)) \xrightarrow[k\to\infty]{} 0$. Suppose also that the functional series $\sum_{j=0}^{\infty}(\varphi^{-1}\psi)^j(\varphi^{-1}(t))$ pointwise converges at any point $t, 0 \le t \le R$, and its sum $S(t)$ is right-continuous at zero.*

Then, there exists a number $N \in \mathbb{N}$, such that for any $k > N$, there is a solution ξ_k of the equation $F_k(x,x) = z_0$, such that the sequence $\{\xi_k\}_{k=1,2,...}$ converges to x_0, and $\rho(x_0, \xi_k) \leq \sum_{j=0}^{\infty}(\varphi^{-1}\psi)^j(\varphi^{-1}(r_{0k})) = S(r_{0k}) \leq R$.

Proof. The proof scheme is similar to that of [13, Theorem 2]. As $r_{0k} = d(z_0, F_k(x_0, x_0)) \xrightarrow[k\to\infty]{} 0$ and the function $S(t)$ is right-continuous at zero, there is a number N such that for any $k, k > N$, it is true that $S(r_{0k}) \leq R$. It follows that all conditions of Theorem 4 hold for any $F_k, k > N$. Then, in accordance with Theorem 4, for any $k, k > N$, there exists $\xi_k \in B(x_0, R)$ such that $F_k(\xi_k, \xi_k) = z_0$ and $\xi_k \xrightarrow[k\to\infty]{} x_0$. More exactly, the following estimation holds: $\rho(x_0, \xi_k) \leq \sum_{j=0}^{\infty}(\varphi^{-1}\psi)^j(\varphi^{-1}(r_{0k})) = S(r_{0k})$. □

Given $x_0 \in X$, $z_0 \in Y$, $R > 0$, $F : B(x_0, R) \times B(x_0, R) \to Y$, the essence of the proof of Theorem 4 above is the constructing, under special conditions (i)–(iv), of approximation sequences $\{x_k\}_{k=0,1,...} \subset X$ and $\{z_k\}_{k=0,1,...} \subset Y$, $z_k = F(x_k, x_k)$, $k = 0, 1, ...$, converging to some $\xi \in B(x_0, R)$ and z_0 respectively, with $F(\xi, \xi) = z_0$. Conditions (i)–(iv) of Theorem 4 (and all the more, the conditions of [13, Theorem 1]) are rather strong and concern the global behaviour of the given mapping F, including its special behaviour along each coordinate. However, a result providing the existence and approximation of a solution of the equation $F(x, x) = z_0$ can be obtained with the use of functionals strictly subordinated to a convergent series, under less stringent conditions on the mapping F. Below we consider the more general problem of the existence and approximation of a solution of the inclusion $z \in F(x, x)$, where $F : X \times X \to BC(Y)$ is a multivalued mapping, and $BC(Y)$ stands for the totality of nonempty bounded closed subsets of Y.

The next theorem is a consequence of Theorem 1 given above.

Theorem 6. *Let $(X, \rho), (Y, d)$ be metric spaces, and let X be complete. Let $x_0 \in X$, $z_0 \in Y$, $R > 0$, and a mapping $F : B(x_0, R) \times B(x_0, R) \to BC(Y)$ be given. Suppose that the multivalued functional $\varphi(x) = \{t \in \mathbb{R}_+ \mid t = d(y, z_0), y \in F(x, x)\}$ is strictly subordinated to series (1), its graph $Gr(\varphi)$ is $\{0\}$-closed, and there is $t_0 \in \varphi(x_0)$, such that $t_0 \leq c_{k_0} \cdot \min\{1, \frac{R}{S_{k_0}}\}$, for some $k_0 \in \mathbb{N}$. Then there exists $\xi \in X$ such that $z_0 \in F(\xi, \xi)$, and $\rho(x_0, \xi) \leq \frac{t_0 \cdot S_{k_0}}{c_{k_0}} \leq R$.*

Proof. The reasonings are quite standart. All conditions of Theorem 1 are fulfilled for the functional φ. Then it follows from Theorem 1, that on the graph $Gr(\varphi)$ a multicascade is defined with nonempty limit set $\mathcal{A}$ which projection $\mathcal{A}_X$ onto X is equal to $Nil(\varphi) = \{x \in B(x_0; R) \mid 0 \in \varphi(x)\} = \{x \in B(x_0; R) \mid z_0 \in F(x, x)\}$. Every trajectory of that multicascade beginning from a pair $(x_0, t_0) \in Gr(\varphi)$, with $t_0 \leq c_{k_0} \cdot \min\{1, \frac{R}{S_{k_0}}\}$, converges to some pair $(\xi, 0) \in Gr(\varphi)$, $\xi \in \mathcal{A}_X$, and $\rho(x_0, \xi) \leq \frac{t_0 \cdot S_{k_0}}{c_{k_0}} \leq R$. □

Theorem 6 implies the following analogue of Theorem 5.

Theorem 7. *Let $(X, \rho), (Y, d)$ be metric spaces, and let X be complete. Let $x_0 \in X, z_0 \in Y, R > 0$, and mappings $F_k : B(x_0, R) \times B(x_0, R) \to BC(Y), k \in \mathbb{N}$, be given. Suppose that all functionals $\varphi_k : B(x_0; R) \to P(\mathbb{R}_+)$, $\varphi_k(x) = \{t \in \mathbb{R}_+ \mid t = d(y, z_0), y \in F_k(x, x)\}$, $k \geq 1$, are strictly subordinated to series (1), their graphs $Gr(\varphi_k)$ are $\{0\}$-closed, and $\sup(\varphi_k(x_0)) \xrightarrow[k\to\infty]{} 0$. Then, for any $q \in \mathbb{N}$, there exists a number $N = N(q) \in \mathbb{N}$ such that for any $k > N$ there is a solution ξ_k of the inclusion $z_0 \in F_k(x, x)$, and $\xi_k \xrightarrow[k\to\infty]{} x_0$. More exactly, $\rho(x_0, \xi_k) \leq \frac{\sup(\varphi_k(x_0)) \cdot S_q}{c_q} \leq R$.*

Proof. As $\sup(\varphi_k(x_0)) \xrightarrow[k\to\infty]{} 0$, then for any $q \in \mathbb{N}$, there is a number $N = N(q) \in \mathbb{N}$ such that for any $k > N$, $\sup(\varphi_k(x_0)) \leq c_q \cdot \min\{1, \frac{R}{S_q}\}$. Then, for any $k > N$, all conditions of Theorem 6 are fulfilled for the mapping F_k and for any $t_0 \in \varphi_k(x_0)$. Consequently, in accordance with Theorem 6, for any $k > N$, there exists $\xi_k \in X$ (may be not unique) such that $z_0 \in F_k(\xi_k, \xi_k)$, and $\rho(x_0, \xi_k) \leq \frac{\sup(\varphi_k(x_0)) \cdot S_q}{c_q} \leq R$, which was to be proved. □

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